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# Landau-Ginzburg Lagrangians of minimal $W$ -models with an integrable perturbation\*

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## Abstract

We construct Landau-Ginzburg Lagrangians for minimal bosonic ( $N = 0$ )  $W$ -models perturbed with the least relevant field, inspired by the theory of  $N = 2$  supersymmetric Landau-Ginzburg Lagrangians. They agree with the Lagrangians for unperturbed models previously found with Zamolodchikov's method. We briefly study their properties, e.g. the perturbation algebra and the soliton structure. We conclude that the known properties of  $N = 2$  solitons (BPS, lines in  $W$  plane, etc.) hold as well. Hence, a connection with a generalized supersymmetric structure of minimal  $W$ -models is conjectured.

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# 1 Introduction

The construction of Landau-Ginzburg Lagrangians for 2d conformal field theories (CFT) is a powerful tool in the study of these theories as a description of the critical behaviour of statistical models. Besides, the case with  $N = 2$  supersymmetry has found wide application in string theory. It has been shown before that Landau-Ginzburg Lagrangians can be constructed for a wide class of 2d CFT, the minimal models of  $W$ -algebras [1]. Those Landau-Ginzburg (LG) Lagrangians describe the multi-critical behaviour of two-dimensional systems with dihedral  $D_n$  symmetry; in particular, they describe the multi-critical behaviour of the integrable IRF models introduced by Jimbo *et al* [2].

The LG Lagrangians for minimal  $W$ -models were obtained by a generalization of the method of Zamolodchikov for the minimal Virasoro models [3]. Thus a composite field structure was given to the algebra of relevant primary fields and a field equation ensued from the truncation of this algebra. However, the complete Landau potential was obtained only for the case of  $W_3$ . For  $W_3$ -models it was shown that there is a perturbation that reproduces the ground state structure of the corresponding IRF model and that this is the maximum possible unfolding of the potential [1, 4]. Although it is possible in principle to obtain similar results for  $n > 3$ , the algebraic methods utilized for  $n = 3$  are hardly generalizable.

The theory of perturbed Landau potentials for 2d CFT with  $N = 2$  supersymmetry is however well developed [5]. When the perturbation is the least relevant field, one has that the superpotential is precisely the fusion potential of  $SU(n)$  WZW theories found by Gepner [6]. Furthermore, the  $SU(n)$  WZW fusion algebra coincides with the algebra of chiral fields of the  $N = 2$  CFT. Unfortunately, the Gepner potential is complex and hence not suitable for the non-supersymmetric case. Nevertheless, one can derive from it a bosonic potential with interesting properties [10]. In the one-variable case ( $n = 2$ ) the  $N = 2$  bosonic potential has the expected properties for a  $N = 0$  potential; namely, if we label the minimal models with  $p = 3, \dots$ , as usual, it has degree  $2(p-1)$  and  $p-1$  minima with zero energy. However, for  $n > 2$  its degree is higher than that required by the structure of the algebra of relevant fields and moreover it does not constitute a well defined real singularity when the perturbation is set to zero. Nevertheless, there exists a remarkable relation between the fusion algebras of the  $SU(n)$  WZW theories and certain field subalgebras of the  $W$ -models that indicates that Gepner's construction is also relevant for  $N = 0$ . We shall rely on it to find perturbed Landau potentials that agree with those obtained with Zamolodchikov's method.

# 2 The algebra of relevant fields of $W$ -models

We will consider only modular diagonal invariant models for which we need only spin zero fields. Therefore, we shall only consider the holomorphic part of fields. Besides, the underlying Lie algebra of the  $W$ -symmetry is always  $A_{n-1}$ . The algebra of relevant

primary fields of the model  $W_{(n)}^p$  has been described before [1]. It consists of two parts: First, the fields of lower dimension,  $\Phi(\boldsymbol{\lambda} \mid \boldsymbol{\lambda})$ , which were called diagonal, with dimension

$$\Delta = \frac{\boldsymbol{\lambda}(\boldsymbol{\lambda} + 2\boldsymbol{\rho})}{2p(p+1)},$$

proportional to the first Casimir of the  $SU(n)$  representation with highest weight  $\boldsymbol{\lambda}$ . They fill a Weyl alcove of level  $k = p - n$ . The second part consists of the fields

$$\Phi(\boldsymbol{\lambda} - \boldsymbol{\alpha} \mid \boldsymbol{\lambda})$$

where  $\boldsymbol{\alpha}$  is a positive root or, in some cases, the sum of two positive roots; these fields are called non-diagonal. They are all generated successively as powers of the elementary fields

$$x_k = \Phi(\boldsymbol{\omega}_k \mid \boldsymbol{\omega}_k), \quad k = 1, \dots, n-1, \quad (1)$$

$$x_{n-k} = \bar{x}_k, \quad (2)$$

with  $\boldsymbol{\omega}_k$  the  $k$ th fundamental weight, according to the standard definition of composite fields [3]. For the diagonal fields one has

$$\Phi(\boldsymbol{\lambda} \mid \boldsymbol{\lambda}) = \prod_{k=1}^{n-1} x_k^{\mu_k}, \quad (3)$$

with  $\mu_k$  the Dynkin labels of  $\boldsymbol{\lambda} = \sum \mu_k \boldsymbol{\omega}_k$ . The subsequent powers of  $x_k$  are identified with non-diagonal fields, though this identification is not as straightforward as (3) [1].

There is a distinguished subalgebra of relevant fields, generated by

$$\epsilon := \Phi(\boldsymbol{\theta} \mid \boldsymbol{\theta}) = x_1 \bar{x}_1, \quad (4)$$

called the thermal subalgebra. Their diagonal fields are

$$\epsilon^l = \Phi(l\boldsymbol{\theta} \mid l\boldsymbol{\theta}), \quad 2l \leq p - n, \quad (5)$$

and their non-diagonal fields

$$\epsilon^l = \Phi((p - n - l)\boldsymbol{\theta} \mid (p - n - l + 1)\boldsymbol{\theta}), \quad p - n < 2l \leq 2(p - n). \quad (6)$$

The least relevant field of the thermal subalgebra,

$$\epsilon^{p-n} = \Phi(\mathbf{0} \mid \boldsymbol{\theta}), \quad \Delta = 1 - \frac{n}{p+1} \lesssim 1, \quad (7)$$

produces the field equation upon multiplication by  $x_1$

$$x_1 (x_1 \bar{x}_1)^{p-n} = \partial^2 x_1. \quad (8)$$

However, the ensuing Lagrangian

$$\mathcal{L} = \partial x_1 \partial \bar{x}_1 + (x_1 \bar{x}_1)^{p-n+1} \quad (9)$$

is incomplete, as can be seen from its having too large symmetry. It is easy to show that a lower degree term with the correct symmetry has been omitted [1],

$$\delta\mathcal{L} = (x_1\bar{x}_1)^{p-n-1} (x_1^2\bar{x}_2 + \bar{x}_1^2x_2). \quad (10)$$

Nevertheless, this Lagrangian is still incomplete. The total Lagrangian can be found for  $W_3$ -models using methods of singularity theory [4].

When there is  $N = 2$  supersymmetry one can identify the fusion rules of a subalgebra of primary fields, that of the chiral fields, with the fusion rules of the affine algebra  $SU(n)_k$  [6]. Thus the critical superpotential turns out to be the quasi-homogeneous part of the fusion potential. This potential,  $\mathcal{W}$  say, is generally complex but there is an associated bosonic potential given by  $V = |\partial_i \mathcal{W}|^2$ , with the property that the extrema of  $\mathcal{W}$  correspond to zero energy ground states of  $V$ . This should be the first candidate of which one could think for the potential of the non supersymmetric  $W$ -models. However, we shall see in the next section that it is not suitable. Nevertheless, we can still identify within the operator algebra fields with the fusion rules of the affine algebra  $SU(n)_k$ , modulo irrelevant fields. These fields are the diagonal fields  $\Phi(\lambda | \lambda)$ . Relying on this fact, we may expect to be able to use the known realization of  $SU(n)_k$  fusion algebras in terms of orthogonal polynomials [7, 6] as well as in the the case with  $N = 2$  supersymmetry.

### 3 Constructing the Landau potentials

First of all, let us see that the bosonic potential  $V = |\partial_i \mathcal{W}|^2$  of  $N = 2$  supersymmetric  $W$ -models is not a good candidate in the case with no supersymmetry. The Gepner fusion potential  $\mathcal{W}$  for  $SU(n)_k$  is of degree  $k + n$ . Hence the bosonic potential  $V$  is of degree  $2(k + n - 1)$ . It does not match the degree of the potentials found before (9), which is  $2(p - n + 1) = 2(k + 1)$  (except when  $n = 2$ ). We saw in a number of cases [1] that the latter potential suffices to produce the correct number of minima under a suitable perturbation. Thus the degree of the former potential being larger means that one should not exclude in principle the existence of further minima, though they would not have zero energy. The possible presence of extra minima would be nevertheless an undesirable feature. The root of the problem is that the form of the unperturbed  $N = 2$  bosonic potentials does not constitute a bona fide real singularity (except for  $n = 2$ ).<sup>\*</sup> In relation to it one could add another reason to discard the  $N = 2$  bosonic potential: Let us assign degree  $k$  to  $x_k$  and  $\bar{x}_k$ ,  $k \leq [n/2]$ . When the perturbation is set to zero this potential becomes inhomogeneous whereas the unperturbed potential given by (9) and (10) is homogeneous. We should look for a perturbed potential whose (quasi-)homogeneous part agrees with (9) and (10).

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<sup>\*</sup>This fact seems to have been overlooked in the literature. Presumably, it is not crucial for the  $N = 2$  case, where one is essentially interested in the chiral ring and hence the holomorphic potential  $\mathcal{W}$ , which is a well defined complex singularity.

Let us recall the reason why the bosonic potential has as zero-energy ground states a set of points corresponding to the Weyl alcove of level  $k$ : They are the solutions of the equations  $\partial_i \mathcal{W} = 0$  [6]. Indeed, one can see that these equations imply the vanishing of the polynomials representing fields at level  $k+1$ . In the language of algebraic geometry one can actually identify (as a category functor) a finitely generated algebra with a set of points. Since we expect precisely the set of points above to be the minima of the perturbed potential for which we are looking, we must also expect that an algebra equivalent to that of  $SU(n)_k$  to appear. It does indeed appear as the fusion algebra of diagonal fields. To better understand their physical rôle we must bring about the connection with integrable IRF models. The ground states of these models are in correspondence with the points of a graph, the mentioned Weyl alcove [2]. It is natural to define a set of order parameters to characterize these ground states. The first candidates are local state probabilities, defined as the expectation value of the projector onto a definite ground state. They are the analog of the point set basis of the algebra, that is, the basis that consists of functions that vanish in all the points except one, for each of them [8, 6]. However, it is preferable to take the linear combinations

$$\Phi^{(a)}(\mathbf{r}) = \sum_{\alpha} \frac{\psi_{\alpha}^{(a)}}{\psi_{\alpha}^{(1)}} \mathcal{P}_{\alpha}(\mathbf{r}),$$

with  $\psi_{\alpha}^{(a)}$  the eigenvectors of the adjacency matrix of the graph and  $\mathcal{P}_{\alpha}$  the local state probability of height  $\alpha$ . Their continuum limit gives the diagonal fields, which are hence understood as the order parameters [9]. (One should exclude the identity operator.) They distinguish between ground states.

From the discussion above we conclude that the conditions for the vanishing of fields at level  $k+1$  must play a decisive role in constructing the potential. Furthermore, we would like the structure of the potential to show clearly that the ground states are the ones given by those conditions. The  $N = 2$  bosonic potential fulfils these requirements but is not suitable. Fortunately, there is another possibility: In place of the derivatives of the Gepner potential we may take the polynomials that represent the fields at level  $k+1$  and define<sup>†</sup>

$$V = \sum_{\lambda^{(k+1)}} |P_{\lambda}(x_i)|^2. \quad (11)$$

The first term occurs for  $\lambda = (k+1)\omega_1$  and has the form

$$\begin{aligned} |P_{(k+1)\omega_1}(x_i)|^2 &= P_{(k+1)\omega_1}(x_i) P_{(k+1)\omega_{n-1}}(x_i) = \\ &= (x_1^{k+1} - k x_2 x_1^{k-1} + \dots)(\bar{x}_1^{k+1} - k \bar{x}_2 \bar{x}_1^{k-1} + \dots) = \\ &= (x_1 \bar{x}_1)^{k+1} - k (x_1 \bar{x}_1)^{k-1} (x_1^2 \bar{x}_2 + \bar{x}_1^2 x_2) + \dots. \end{aligned} \quad (12)$$

Hence it agrees with (9) and (10). All the other terms associated to the weights  $\omega_1$  and  $\omega_{n-1}$  only, namely,  $\lambda = (k+1-j)\omega_1 + j\omega_{n-1}$ ,  $j = 1, \dots, k$ , yield a similar result

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<sup>†</sup>These polynomials are conjugates of one another due to the reflection symmetry of the  $A_{n-1}$  Dynkin diagrams; hence the sum actually runs over half the weights  $\lambda^{(k+1)}$ —or half plus one.

(see appendix A),

$$|P_{\lambda}(x_i)|^2 = (x_1 \bar{x}_1)^{k+1} - (k-1)(x_1 \bar{x}_1)^{k-1} (x_1^2 \bar{x}_2 + \bar{x}_1^2 x_2) + \dots \quad (13)$$

The remaining terms in (11) have lower degree in  $x_1, \bar{x}_1$ . This can be seen by noticing that the highest degree monomial in  $P_{\lambda}(x_i)$  is given by (3) with Dynkin labels such that  $\sum \mu_i = k+1$  at level  $k+1$ . When  $n=3$  then  $x_{n-1} = x_2$  and the remaining terms do not appear, leaving just (12) and (13) with  $x_2 = \bar{x}_1$ .

We may question in general if all the terms in (11) are actually necessary. It may well happen that the vanishing of a subset of the polynomials at level  $k+1$  suffices to imply the vanishing of them all. We shall see next that this is indeed so, although the subset of polynomials  $\lambda = (k-j)\omega_1 + j\omega_{n-1}$  is not the right one. We already know that the vanishing of the derivatives of the Gepner potential,

$$\partial_i \mathcal{W} = P((k+n-i)\omega_1), \quad i = 1, \dots, n-1,$$

implies the vanishing of all the polynomials at level  $k+1$ . One can prove this in a constructive way, making combinations of  $P((k+n-i)\omega_1)$  to produce those polynomials. The first one is naturally  $P((k+1)\omega_1)$ . Multiplying it by  $x_1$  we obtain

$$x_1 P((k+1)\omega_1) = P((k+2)\omega_1) + P(k\omega_1 + \omega_2), \quad (14)$$

whereby we deduce the vanishing of  $P(k\omega_1 + \omega_2)$ . Continuing this procedure we can deduce the vanishing of all polynomials of the form  $P(k\omega_1 + \omega_i)$ ,  $i = 1, \dots, n-1$ , (see appendix B). From them one can proceed with the rest of the polynomials at level  $k+1$ . Conversely, the vanishing of  $P(k\omega_1 + \omega_i)$ ,  $i = 1, \dots, n-1$ , implies that of the polynomials  $P((k+n-i)\omega_1)$ . Thus we conclude that the adequate subset of these polynomials to include in the potential is precisely  $P(k\omega_1 + \omega_i)$ ,  $i = 1, \dots, n-1$ . Therefore, the potential (11) must include in its highest degree part terms like  $|x_1^k x_i|^2$ , contradicting somehow the simple form (9) obtained by Zamolodchikov's method. However, their presence can be understood in this context as follows: The simple form (9) was obtained from the field equation (8) but one can and must consider other field equations coming from the product of  $\epsilon^k$  with other elementary fields,

$$x_i (x_1 \bar{x}_1)^{p-n} = \partial^2 x_i. \quad (15)$$

They give rise to the desired terms. Henceforth we consider in (11) only the terms  $|P(k\omega_1 + \omega_i)|^2$ ,  $i = 1, \dots, n-1$ .

Once we have the perturbed Landau potentials, it is straightforward to obtain the critical potential: One just has to put a coupling constant before the term  $\epsilon^k = (x_1 \bar{x}_1)^k$  in the expansion of the potential and its adequate powers before the remaining terms according to dimensional analysis. Then one lets the constant go to zero. These singular potentials are probably equivalent to some of the real singularities listed in the literature, though their complexity greatly hinders any analysis to that effect.

## 4 Further properties of perturbed Landau potentials

It is clear that the Landau potential (11) has the required minima by construction. We may wonder about other properties, e.g., the other extrema or the possible solitons connecting those minima. The existence of maxima and saddle points is related with the presence of non-diagonal relevant fields: In the algebraic-geometric picture presented in the previous section extra points correspond to extra generators in the algebra. The total algebra is defined by the relations  $\partial_x V = \partial_y V = 0$ . The remaining generators can in general be obtained as in [4]; namely, one can use the relations to eliminate all but a finite number of polynomials in the  $x_i$ . To just find their number it may be sufficient to consider the solutions of the equations of motion [4]. However, the problem in general is very complicated. In any event, these non-diagonal fields are not associated to any physical phase transition (they are not order parameters), although they are significant for the topology of the potential.

As regards the question about solitons, it has been conveniently solved for theories with  $N = 2$  supersymmetry: The soliton spectrum is constituted by elementary solitons that interpolate between neighbouring minima and whose energy saturates the Bogomolny bound [10]. This structure nicely agrees with what one expects from the connection with solvable lattice IRF models and affine Toda field theories [11]. Moreover, the soliton  $S$ -matrices are tensor products of  $N = 0$   $S$ -matrices with a fixed  $N = 2$   $S$ -matrix. Therefore, we should expect similar properties for both  $N = 2$  and  $N = 0$  solitons.

One can see that the argument in [10] still works for the present bosonic potentials, due to their close relation with those of theories with  $N = 2$  supersymmetry. The complete form of the bosonic  $N = 2$  potential includes a real Kähler metric  $g$ ,

$$V = (g^{-1})^{\mathbf{x}\bar{\mathbf{x}}} |\partial_{\mathbf{x}} \mathcal{W}|^2. \quad (16)$$

The proof of the essential properties of  $N = 2$  solitons goes through irrespective of the form of that metric. Now we can see that the potential proposed in the previous section can be put in this form. This is due to the fact that the polynomials  $P(k\omega_1 + \omega_i)$  are linear combinations of derivatives of the Gepner potential  $\mathcal{W}$  with  $x_i$ -dependent coefficients (14)(see appendix B for the complete expressions). Hence, the potential can be written in the form (16) for some polynomial  $g^{-1}$ . The metric is not Kähler (there is no reason why it should be) but it does not matter in a purely bosonic theory. Furthermore, the determinant of  $g^{-1}$  is 1 (appendix B), which implies that  $g$  is also polynomial. Hence, the kinetic term of the Landau-Ginzburg Lagrangian,  $g\mathbf{x}\bar{\mathbf{x}} \partial_{\mathbf{x}} \partial_{\bar{\mathbf{x}}}$ , is equivalent to its basic part,  $\partial_{\mathbf{x}} \partial_{\bar{\mathbf{x}}}$ , modulo a finite number of irrelevant fields.

The peculiar properties of  $N = 2$  solitons are directly related with the existence of the supersymmetries. Then it may seem rather unexpected to find similar properties in a purely bosonic theory. Its ultimate reason probably lies in the presence in these theories of quantum affine symmetry. This powerful symmetry is the reason for their

integrability. Besides, it has been shown that it can be understood as a sort of fractional supersymmetry [12]. Moreover, this symmetry can form the basis of a generalized chiral-ring structure and hence Landau-Ginzburg Lagrangians. A preliminary attempt in this direction was presented for  $SU(2)$  in [13]. The elementary field considered there was also the most relevant field. The chiral algebra would correspond to the diagonal fields here. However, the ensuing Lagrangians are the straightforward generalization of the  $N = 2$  ones, which thence were formulated in terms of pseudo-Grassmann variables, and there was made no attempt to construct from them the purely bosonic Landau-Ginzburg Lagrangians. One can speculate that the Landau-Ginzburg Lagrangians constructed here are suitable candidates to fit in a chiral-ring picture once generalized to  $n > 2$ .

## 5 Conclusion

We have combined Zamolodchikov's method with known results for perturbed  $N = 2$  potentials in order to construct perturbed  $N = 0$  potentials. To make them agree we propose an alternative to the bosonic  $N = 2$  potential which reduces its degree to the correct value; namely, we propose to select a different set of vanishing polynomials that also produce truncation at level  $k$ . Finally, it was shown that this modification only amounts to the introduction of a non-trivial hermitian (though non-Kähler) metric in the bosonic  $N = 2$  potential. Therefore, the essential properties of these potentials, in particular as regards to solitons, are preserved.

Our bosonic potentials coincide with the bosonic  $N = 2$  potentials for the one-variable ( $n = 2$ ) case. Moreover, in the  $n = 3$  case the unperturbed part of the potentials is that previously found in [4]. The potentials for  $n > 3$  constitute the natural generalization of those previous results. Since these potentials represent  $W$ -models perturbed by the least relevant field, one can produce the corresponding multicritical second-order phase transition by turning off that perturbation. Below the transition the coupling is negative and we have the full unfolding of minima. Above the transition the coupling is positive and the perturbation  $\epsilon^k = (x_1 \bar{x}_1)^k$  becomes irrelevant, that is, turns into the new potential. Thus the coupling constant of this perturbation must be associated to the temperature, hence reproducing the phase transition between regimes III and IV of the Jimbo *et al* IRF models [2].

The construction relies on the existence in the  $W$ -models of a  $SU(n)_k$  algebra of diagonal fields and hence of a Gepner potential. According to results for  $n = 2$  [13] this algebra could be further given the structure of a chiral algebra in an analogous sense as for  $N = 2$ . This possibility would suggest that the Gepner potential itself has a physical rôle in the bosonic theories and methods of complex singularity theory similar to the ones used in  $N = 2$  (chiral rings) might be also applicable to  $N = 0$ .

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## A The polynomials in the $(\omega_1, \omega_{n-1})$ plane

Let us find the form of the polynomials  $P(\lambda = (k+1-j)\omega_1 + j\omega_{n-1})$ ,  $j = 1, \dots, k$ . According to [6] they are given by the Giambelli formula

$$P((k+1-j)\omega_1 + j\omega_{n-1}) = \underbrace{[1, \dots, 1, n-1, \dots, n-1]}_{k+1-j} =$$

$$= \left| \begin{array}{ccccccccc} x_1 & 1 & 0 & \cdots & \cdots & & & & 0 \\ x_2 & \ddots & \ddots & \ddots & & & & & \\ 0 & \ddots & \ddots & \ddots & \ddots & & & & \\ & \ddots & \ddots & \ddots & 1 & 0 & & & \vdots \\ \vdots & & \ddots & x_2 & x_1 & 1 & 0 & & \\ & & & 0 & 1 & x_{n-1} & x_{n-2} & \ddots & \vdots \\ \vdots & & & 0 & 1 & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & \cdots & & 0 & 1 & x_{n-2} & \\ \end{array} \right| =$$

$$x_1^{k+1-j} x_{n-1}^j - (k-j)x_1^{k-j-1} x_2 x_{n-1}^j - (j-1)x_1^{k+1-j} x_{n-2} x_{n-1}^{j-2} + \dots \quad (17)$$

Hence

$$|P((k+1-j)\omega_1 + j\omega_{n-1})|^2 = (x_1 \bar{x}_1)^{k+1} - (k-1) (x_1 \bar{x}_1)^{k-1} (x_1^2 \bar{x}_2 + \bar{x}_1^2 x_2) + \dots \quad (18)$$

## B Relation between $P(k\omega_1 + \omega_i)$ and $\partial_j \mathcal{W}$

Here we shall show how to express the polynomials  $P(k\omega_1 + \omega_i)$ ,  $i = 1, \dots, n-1$ , in terms of the polynomials  $\partial_j \mathcal{W} = P((k+n-j)\omega_1)$ ,  $j = 1, \dots, n-1$ . We need to generalize eq. (14) to  $i > 2$ . For this we take  $P((k+j)\omega_1)$  ( $j$  immaterial) and multiply it by an arbitrary  $x_i$ ,  $i = 1, \dots, n-1$ . In this product shall appear any  $P(\lambda)$  such that  $\lambda$  belongs to the Clebsch-Gordan decomposition of  $(k+j)\omega_1 \oplus \omega_i$  (in an obvious notation), namely,  $\lambda = (k+j)\omega_1 + \omega_i$  and  $\lambda = (k+j)\omega_1$  plus a Weyl transform of  $\omega_i$ . All these transforms except the first take  $(k+j)\omega_1$  out of the fundamental Weyl dominion and therefore yield a null result. The first transform is given by  $\omega_i = \epsilon_1 + \dots + \epsilon_i \rightarrow \epsilon_2 + \dots + \epsilon_{i+1}$  with  $\{\epsilon_i\}_{i=1}^n$  the projection of an orthonormal

basis of  $\mathbb{R}^n$ ,  $\{\mathbf{e}_i\}_{i=1}^n$ , orthogonal to  $\sum_{i=1}^n \mathbf{e}_i$ . Since  $\epsilon_2 + \cdots + \epsilon_{i+1} = -\omega_1 + \omega_{i+1}$  we finally find that

$$x_i P((k+j)\omega_1) = P((k+j)\omega_1 + \omega_i) + P((k+j-1)\omega_1 + \omega_{i+1}). \quad (19)$$

If  $i = j = 1$  we have eq. (14). When  $i + j = 3$  we have two equations and likewise onwards. Adding the  $m - 1$  equations for  $i + j = m$  with alternate signs, all  $P((k+j-1)\omega_1 + \omega_{i+1})$  cancel except the last,  $P(k\omega_1 + \omega_m)$ . Solving for it for every  $m = 2, \dots, n - 1$  we obtain the set of relations

$$\begin{aligned} P(k\omega_1 + \omega_m) = x_{m-1} P((k+1)\omega_1) - x_{m-2} P((k+2)\omega_1) + x_{m-3} P((k+3)\omega_1) - \\ \cdots \pm P((k+m)\omega_1) \end{aligned} \quad (20)$$

or, in matrix form,

$$\begin{bmatrix} P((k+1)\omega_1) \\ P(k\omega_1 + \omega_2) \\ P(k\omega_1 + \omega_3) \\ \vdots \\ P(k\omega_1 + \omega_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_1 & -1 & \ddots & \vdots \\ x_2 & -x_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ x_{n-2} & -x_{n-3} & \cdots & \pm 1 \end{bmatrix} \begin{bmatrix} P((k+1)\omega_1) \\ P((k+2)\omega_1) \\ P((k+3)\omega_1) \\ \vdots \\ P((k+n-1)\omega_1) \end{bmatrix}. \quad (21)$$

If we call the transformation matrix  $T$ , it is immediate that  $\det T = \pm 1$ .

The potential (11) can thence be expressed as

$$V = \sum_{\lambda^{(k+1)}} |P_\lambda(x_i)|^2 = \sum_{\mu\bar{\mu}} T_{\lambda\mu} \overline{T_{\lambda\mu}} |P_\mu(x_i)|^2, \quad (22)$$

where  $\lambda = k\omega_1 + \omega_m$  and  $\mu = (k+m)\omega_1$  ( $m = 1, \dots, n - 1$ ). Therefore,

$$V = (g^{-1}) \mathbf{x} \bar{\mathbf{x}} |\partial \mathbf{x} \mathcal{W}|^2 \quad (23)$$

with

$$(g^{-1}) \mathbf{x} \bar{\mathbf{x}} = T \overline{T} \delta \mathbf{x} \bar{\mathbf{x}}. \quad (24)$$

Note that  $\det g^{-1} = 1$ .

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